CS 4510

Rice's Theorem

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9.1 Properties

Let $\mathcal{M}$ be the set of all Turing machines. A property is a partition of the set of all Turing machines. Each Turing machine either has, or hasn't the property. If $L(M_1) = L(M_2)$ then $\langle M_1 \rangle, \langle M_2 \rangle$ are both in P or both not in P.

A nontrivial property is one where $P \neq \emptyset$, and $P \neq \mathbb{M}$. For example, $P = \{\langle M \rangle \mid 0 \in L(M)\}$ is nontrivial, since there exists at least one machine in $P \setminus \mathcal{M}$, and there is atleast one machine in P. As another example, $P = \{\langle M \rangle \mid L(M) \subseteq \Sigma^*\}$ is a trivial property, since $P = \mathcal{M}$.

9.2 Rice's Theorem

Theorem 9.1 *Every nontrivial property is undecidable.*

Before going further, I would like to comment on what this theorem is really saying. Properties are about semantics, not syntatics. For example, the set of Turing machines which has exactly n states is a syntatic property. You may simply just count the states. The set of Turing machines whos language is equally recognizable by a machine with exactly n states is a *semantic* property. In general, you cannot learn any semantic properties of the language a machine recognizes only given the encoding of the machine. This is a theorem not about machines, but about the recognizable languages. Determining if a machine $\langle M \rangle$ has some property is logically equivalent to determining if $\langle M \rangle \in P$.

9.2.1 Proof

Let P be some nontrivial property, and let $L(P) = \{L(M) \mid \langle M \rangle \in P\}$. Then there exists machines M_0, M_1 such that $M_0 \in P$ and $M_1 \notin P$. Assume to the contrary that P is decidable. Without loss of generality assume $\emptyset \notin L(P)$. We construct a decider for A_{TM}.

On input $\langle M, w \rangle$:
    Construct$\langle M' \rangle$ hardcoded from M, w
    accept $\iff$ $\langle M' \rangle \in P$

M': On input x:
        Run M on w
        if it accepts:
          run M_0 on x
            if it accepts:
              accept
        else:
          reject

Notice that M accepts $w \iff L(M') = L(M_0) \in L(P)$ and that $L(M') = \emptyset \notin L(P)$ otherwise. So we have constructed a decider for A_TM, a contradiction, so P must be undecidable. But P is any nontrivial property! So all nontrivial properties are undecidable.

9.3 Example

Ye be warned! Rice's theorem is excluded from the Sipser book because its too dangerous[1]. On exams, students will often try to use Rice's theorem incorrectly to avoid having to a reduction. We have a safety process. Lets prove that $L = \{\langle M\rangle \mid L(M) \text{ is finite }\}$ is undecidable. First, check, is this property nontrivial? Yes, I can construct two Turing machines for $\emptyset, \Sigma^*$ respectively. Second check, is this a property? Is it true that if $L(M_1) = L(M_2)$ that $\langle M_1\rangle \in P \iff \langle M_2\rangle \in P$? Yes, if $L(M_1) = L(M_2)$, they are either both finite or not. A set can't be finite and infinite at the same time, so it is a property. The two checks are clear, and we are free to apply. "By Rice's theorem, this language is undecidable."

Rice's theorem can make proofs really easy, this was super short, and applications of the theorem are about this easy. You just have to be careful.

9.4 Problems

1. Why can we assume in the proof $\emptyset \notin L(P)$? I said WLOG, but why?
2. Lets the problems you have seen before, but with Rice's Theorem. Prove four of the languages are undecidable via Rice's theorem
   (a) $E_{TM} = \{\langle M\rangle \mid L(M) = \emptyset\}$[2]
   (b) $REGULAR_{TM} = \{\langle M\rangle \mid L(M) = \text{ is a regular language}\}$[3]
   (c) $E_{TM} = \{\langle M\rangle \mid L(M) = \emptyset\}$
   (d) $T = \{\langle M\rangle \mid w \in L(M) \iff w^R \in L(M)\}$[4]
   (e) $INFINITE_{TM} = \{\langle M\rangle \mid |L(M)| \text{ is infinite}\}$[5]
   (f) $\{\langle M\rangle \mid 1011 \in L(M)\}$[6]
   (g) $ALL_{TM} = \{\langle M\rangle \mid L(M) = \Sigma^*\}$[7]
   (h) $E_{TM} = \{\langle M\rangle \mid |L(M)| \geq 1000\}$[8]
3. Why is it necessary that P be non trivial?
4. Why is it necessary that if $L(M_1) = L(M_2)$ implies $\langle M_1\rangle \in P \iff \langle M_2\rangle \in P$?

[1]Thats just my theory
[2]Theorem 5.2
[3]Theorem 5.3
[4]5.25
[5]5.18
[6]5.18
[7]5.18
[8]exam 3