Cyclic Operations on a Puzzle Cube

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1 Introduction

Oftentimes when someone wants to fiddle with a $3 \times 3 \times 3$ puzzle cube, but does not want to really scramble it, a pattern that is often done is U R U' R'. If this is repeated six times, the cube will return back to its identity (solved) state. Another immediate pattern that when repeated, takes you back to the identity state is to pick any face, and simply rotate it four times. Testing other operations and with enough patience, they all seem to return to the identity state. The conjecture immediately comes to mind, "Are all operations on a puzzle cube cyclic?" As in, will every sequence of moves, repeatedly acted upon a cube in its identity state, eventually return a cube to its identity state? The answer is yes and we will prove it in this paper.

2 Background

Before we can get to the proof we need to make several assertions about what a puzzle cube is and what you can legally with a puzzle cube. We define operations to be a sequence of physical moves or other operations.

- **Operations are invertible** - That is to say, If you can get from some state $A \rightarrow B$ with operations $a_1, a_2, ..., a_k$, then you can equally get from state $B \rightarrow A$ with operations $a_k^{-1}, a_{k-1}^{-1}, ..., a_1^{-1}$. We define inverse operations as $a_i \cdot a_i^{-1} = id$, the identity operation. Which does nothing.
- **Operations are deterministic** - If you can get from state $A \rightarrow B$ with some operation a_i, this operation will always hold. If at some other time you are at state A, performing operation a_i will always take you to state B.
- **Puzzle cubes are finite** - There exist only a finite amount of states in our puzzle cube. For example it is known that there are $8! \times 3^7 \times 12! \times 2^{10}$ [1] possible states for a $3 \times 3 \times 3$ puzzle cube.

It is easier to not think of puzzle cubes in terms of the algebraic structure of their pieces, or their operations. Rather it is better to think of the structure of their different states and how they relate to each other under the operations.

3 Proof

Assume to the contrary that there exists some operation a such that when performed upon a puzzle cube, the cube will never return to its identity state. We have two possible cases, the states go on infinitely, or after some moves, are stuck in a loop not involving the identity state. Since puzzle cubes we are working with are finite we may assume that it must be the second case

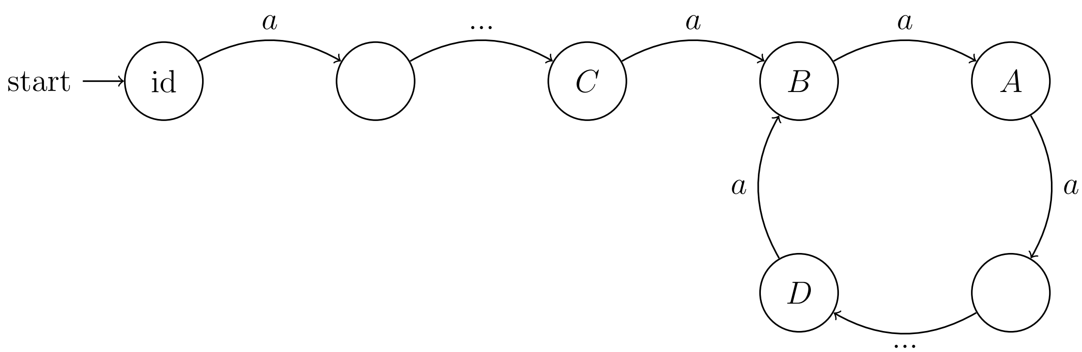
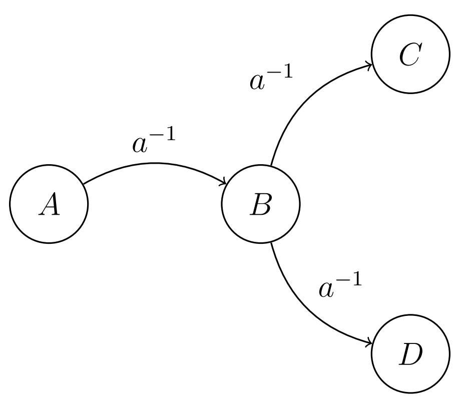


Figure 1: An example of what we describe. Several states are propagated through by a, before becoming stuck in some finite loop by a.

Consider the states we have described, but under the operation a^{-1}. a took $C \to B, B \to A$ and $D \to B$. So operation a^{-1} will take us $A \to B, B \to C, B \to D$



We see our immediate contradiction. State B, under operation a^{-1} takes us to two states, namely C, D simultaneously. This violates the fact that our puzzle cube is deterministic. Therefore no such case can exist, and we see that all operations, repeated enough will eventually propagate back to the identity state.

4 Remarks

Notice that this proof relies on nothing involving the actual geometry of the puzzle cube itself. The only thing that this relies on is that operations are invertible and deterministic, and the entire puzzle cube itself is finite. This works for any size puzzle cube, along with other shapes, like the tetrahedron, octohedron and others.

References

[1] Martin Schőnert, *Analyzing Rubiks cube with GAP*: the permutation group of Rubik's Cube is examined with GAP computer algebra system. http://www.gap-system.org/Doc/Examples/rubik.html